

Total Edge Irregularity Strength of Graph $L_3 \odot N_m$

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Abstract:

An edge irregular total k -labeling of a simple graph G is a labeling that assigns positive integers to its vertices and edges such that the weight of every edge, defined as the sum of the labels of the edge and its two incident vertices, is distinct. The smallest integer k that allows such a labeling is called the total edge irregularity strength, denoted by $tes(G)$. In this paper, we study the total edge irregularity strength of the corona product of a ladder graph L_3 and a null graph N_m , denoted by $L_3 \odot N_m$. By applying constructive labeling and analyzing the resulting edge weights, we show that all edges can be assigned distinct weights. From **Theorem 1**, it is obtained that $tes(L_3 \odot N_m) = 2m + 3$. This result contributes to the development of graph labeling theory and can be extended to larger ladder graphs for further applications, including cryptography and network security.

Keywords: Corona product, Graph Labeling, Irregularity Strength, Null Graph, Ladder Graph

Introduction

A graph is a mathematical structure consisting of a set of vertices and a set of edges that connect pairs of vertices. Formally, a graph is denoted as an ordered pair $G = (V, E)$, where $V(G)$ is a non-empty set of vertices and $E(G) \subseteq \{ \{u, v\} \mid u, v \in V(G), u \neq v \}$ is a set of unordered pairs of distinct vertices, known as edges. When a graph contains no multiple edges or loops, it is called a simple graph.

The concept of Total Edge Irregularity Strength (TES) was first systematically studied by Ahmad and Bača (2009). It refers to a specific type of total labeling, where a function $f: V(G) \cup E(G) \rightarrow \{1, 2, \dots, k\}$ assigns positive integers to both vertices and edges such that the edge weight $w(uv) = f(u) + f(v) + f(uv)$ is unique for every edge $uv \in E(G)$. The smallest integer k for which such a labeling exists is called the total edge irregularity strength, denoted by $tes(G)$. Ahmad and Bača (2009) established a general lower bound for TES as:

$$tes(G) = \max \left\{ \left\lceil \frac{E(G) + 2}{3} \right\rceil, \left\lceil \frac{\Delta(G) + 1}{2} \right\rceil \right\}$$

where $\Delta(G)$ is the maximum degree of the graph G .



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Subsequent research has expanded the study of TES across a wide range of graph classes. For example, Siddiqui et al. (2020) investigated TES in product graphs formed from the Cartesian product of two paths, while Salama (2022) analyzed TES in heptagonal snake graphs through algorithmic approaches. Other notable contributions include Khotimah and Susanti (2020), who studied TES on double fan graphs, and Ratnasari et al. (2021), who focused on book graphs of quadruplet and quintuplet types. Huda and Susanti (2023) examined TES on triangular grid graphs and their derived structures, whereas Ramalakshmi and Kathiresan (2021) characterized TES in several cycle-related graphs. Additionally, Teresa and Visithra (2022) explored TES in various plane graphs. In addition, Emilet et al. (2024) introduced the concept of total face irregularity strength for certain planar graphs, Khan et al. (2025) investigated the edge-irregular reflexive strength of non-planar graphs, and Pandiaraj et al. (2025) studied the (α, β) -total edge irregularity strength of some graph classes, showing the continued development and diversification of irregularity strength research in recent years.

However, no previous research has explicitly determined the TES for the corona product of the ladder graph L_3 and the null graph N_m , where the null graph consists of m isolated vertices without any edges. The corona operation between L_3 and N_m results in a graph where each vertex of L_3 is connected to m new vertices, forming a distinctive central-peripheral structure. This graph's structure opens up new opportunities for investigation, particularly concerning total edge irregular labeling.

This study aims to determine the Total Edge Irregularity Strength of the graph formed by the corona product of L_3 and N_m , and to construct a valid labeling scheme that satisfies the total edge irregularity condition. The results are expected to broaden the scope of graph labeling theory and contribute to the understanding of TES in graphs resulting from product operations.

Research Methods

This study began with a comprehensive literature review to establish the theoretical background of Total Edge Irregularity Strength (TES) and the techniques previously used in related works. The focus was then placed on the corona product of the ladder graph L_3 and the null graph N_m , denoted by $L_3 \odot N_m$. Manual simulations for small values of m were conducted by constructing total labelings and verifying whether all induced edge weights were distinct. From these simulations, a consistent labeling pattern was identified, which led to the conjecture that $Tes(L_3 \odot N_m) = 2m + 3$. The conjecture was confirmed for several small instances of m , and a constructive proof was subsequently developed by defining a systematic labeling function for both the vertices and edges. The resulting weight function was shown to produce pairwise distinct edge weights, and a separation argument established that no smaller value of

k could satisfy the condition. Hence, it was formally concluded that $tes(L_3 \odot N_m) = 2m + 3$.

Results and Discussions

This section presents the results of the study on total edge irregular labeling of the corona product between the ladder graph L_3 dan null graph N_m .

The ladder graph $L_3 = (V(L_3), E(L_3))$ corresponds to the ladder graph L_n for $n = 3$, such that $V(L_3) = \{u_i, v_i | 1 \leq i \leq 3\}$ and $E(L_3) = \{(u_i, u_{i+1}), (v_i, v_{i+1}) | 1 \leq i \leq 2\} \cup \{(u_i, v_i) | 1 \leq i \leq 3\}$. Meanwhile, the null graph $N_m = (V(N_m), E(N_m))$ is defined as a graph with $V(N_m) = \{p_j | 1 \leq j \leq m\}$ and $E(N_m) = \emptyset$, meaning it contains m isolated vertices with no edges.

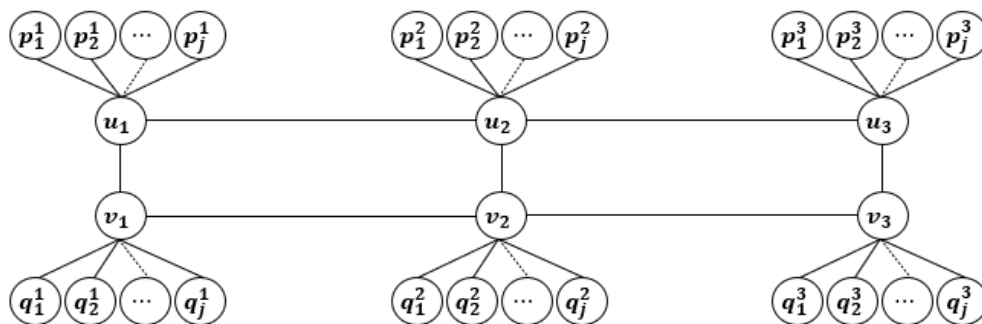
The corona product $L_3 \odot N_m$ results in a graph with vertex set

$$V(L_3 \odot N_m) = V(L_3) \cup \bigcup_{i=1}^3 p_j^i \cup q_j^i$$

for $1 \leq j \leq m$, where p_j^i, q_j^i represent the i -th copies of the null graph N_m connected to vertices $u_i, v_i \in V(L_3)$, respectively. The edge set is given by

$$E(L_3 \odot N_m) = E(L_3) \cup \bigcup_{i=1}^3 \left(\bigcup_{j=1}^m \{(u_i, p_j^i), (v_i, q_j^i)\} \right).$$

Consequently, the total number of vertices is $|V(L_3 \odot N_m)| = 6(m + 1)$, the number of edges is $|E(L_3 \odot N_m)| = 7 + 6m$, and the maximum degree is $\Delta(L_3 \odot N_m) = m + 3$. An illustration of the graph $L_3 \odot N_m$ is provided in Picture 1



Picture 1. Graph $L_3 \odot N_m$

Teorema 1

For every positive integer m , the total edge irregularity strength of the corona product graph $L_3 \odot N_m$ is given by

$$tes(L_3 \odot N_m) = 2m + 3$$

Proof.

We first show that $k \geq 2m + 3$. We know that $|E(L_3 \odot N_m)| = 7 + 6m$ and $\Delta(L_3 \odot N_m) = m + 3$. According to the general lower bound for the total edge irregularity strength of a graph, we have:

$$\begin{aligned} tes(G) &\geq \max \left\{ \left\lceil \frac{E(G) + 2}{3} \right\rceil, \left\lceil \frac{\Delta(G) + 1}{2} \right\rceil \right\} \\ tes(L_3 \odot N_m) &\geq \max \left\{ \left\lceil \frac{(7 + 6m) + 2}{3} \right\rceil, \left\lceil \frac{(3 + m) + 1}{2} \right\rceil \right\} \\ tes(L_3 \odot N_m) &\geq \max \left\{ \left\lceil \frac{6m + 9}{3} \right\rceil, \left\lceil \frac{m + 4}{2} \right\rceil \right\} \\ tes(L_3 \odot N_m) &\geq 2m + 3 \end{aligned}$$

Thus, $k \geq 2m + 3$ is established.

Next, we show that $k \leq 2m + 3$. by constructing a labeling function $f: V(L_3 \odot N_m) \cup E(L_3 \odot N_m) \rightarrow \{1, 2, \dots, 2m + 3\}$ such that all edge weights are distinct. Define the vertex labels as follows, for $1 \leq i \leq 3$ and $1 \leq j \leq m$:

$$\begin{aligned} f(u_i) &= i \\ f(p_j^i) &= 1 + (i - 1)m \\ f(v_i) &= 2m + 4 - i \\ f(q_j^i) &= m(3 - i) + 3. \end{aligned}$$

Define the edge labels as:

$$\begin{aligned} f(u_1u_2) &= m \\ f(u_2u_3) &= 2m - 1 \\ f(u_iv_i) &= m + i \\ f(v_1v_2) &= m + 4 \\ f(v_2v_3) &= 5 \\ f(u_ip_j^i) &= j \end{aligned}$$

$$f(v_i q_j^i) = 2m - j + 4.$$

Using the defined vertex and edge labels, the edge weights are computed according to the formula $w(uv) = f(u) + f(v) + f(uv)$, for all $u, v \in V(L_3 \odot N_m)$. The computed edge weights are as follows:

$$w(u_1 u_2) = m + 3$$

$$w(u_2 u_3) = 2(2 + m)$$

$$w(u_i v_i) = i + 3m + 4$$

$$w(v_1 v_2) = 5m + 9$$

$$w(v_2 v_3) = 4(m + 2)$$

$$w(u_i p_j^i) = i + j + 1 + (i - 1)m$$

$$w(v_i q_j^i) = 7m + 11 - j - i(1 + m).$$

From the calculations, it is evident that all edge weights are distinct when using labels from the set $\{1, 2, \dots, 2m + 3\}$. Hence, $k \leq 2m + 3$. Therefore, the labeling function satisfies the conditions for total edge irregular labeling, and we conclude that $tes(L_3 \odot N_m) = 2m + 3$. ■

The result of Theorem 1 confirms that the total edge irregularity strength of $L_3 \odot N_m$ is $2m + 3$. This demonstrates that the proposed labeling construction successfully ensures all edge weights are distinct, thereby satisfying the total edge irregularity condition. This finding serves as the basis for drawing the main conclusion of the study.

Conclusions and Suggestions

This study has explored the total edge irregularity strength (TES) of the corona product of the ladder graph L_3 and the null graph N_m , resulting in the graph $L_3 \odot N_m$. Through a constructive labeling approach, it was shown that all edge weights can be made distinct using positive integers from the set $\{1, 2, \dots, 2m + 3\}$, thereby establishing that $tes(L_3 \odot N_m) = 2m + 3$. This result confirms that the labeling scheme satisfies the condition for total edge irregular labeling and provides a precise value for the TES of this particular graph structure.

Building on this result, future research is encouraged to generalize the approach to corona products involving $L_n \odot N_m$ for arbitrary values of n , as increasing the size and complexity of the base graph may introduce new challenges in constructing valid labelings. Moreover, due to the nature of edge irregular labeling, where uniqueness and non-repetition are essential. The findings of this study may also contribute to

applied areas such as cryptography, particularly in designing secure schemes for encoding, authentication, and network identification using labeled graph structures.

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