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Interpreting the ε–δ Definition: Hermeneutic Insights into Students' Ways of Thinking

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***Abstract*:**
This study aims to reveal the hermeneutic insight in the formation of students' ways of thinking (WoT) toward the precise meaning of the limit concept in differential calculus. Formally, the concept of limit is defined through the relationship between ε (epsilon) and δ (delta), namely that for every $\varepsilon > 0$, there exists a $\delta > 0$ such that if $0 < |x - a| < \delta$, then $|f(x) - L| < \varepsilon$, expressing a precise dependency between function values and domain values. However, students often interpret this definition in ways that do not fully capture its relational meaning. This qualitative case study involved 28 undergraduate students and used written tasks, interviews, and classroom observations to collect data. The analysis integrates the WoT framework, which consists of empirical, procedural, and theoretical ways of thinking, with a hermeneutic perspective to explore how students construct meaning. The findings show that most students' reasoning is dominated by empirical and procedural ways of thinking of the ε–δ, reflected in substitution, approximation, and symbolic manipulation. From a hermeneutic perspective, these patterns indicate that students interpret limit within restricted interpretative horizons. Only a few students demonstrate theoretical reasoning, suggesting an emerging integration between prior understanding and formal structure. These findings suggest that students' difficulties with limits are not only cognitive but also interpretative, highlighting the importance of supporting meaning-making in calculus learning.

Keywords: Hermeneutic Perspective, Ways of Thinking, Limit Definition, Epsilon-Delta

Introduction

The concept of limit constitutes one of the most essential foundations of differential calculus and simultaneously serves as an entry point to mathematical rigor for university students (Tall & Vinner, 2018; Hitt & Dufour, 2021; Mahadewsing, Getrouw & Calor, 2024; Yoon et al., 2021). Through the study of limits, students are introduced to a precise mode of thinking that underpins their understanding of derivatives, continuity, and integrals. However, numerous studies have shown that the concept of limit frequently becomes a source of misconceptions and profound conceptual difficulties (Sierpinska, 1987; Cottrill et al., 1996; Bezuidenhout, 2010; Gucler, 2013). Most students tend to understand limits intuitively as a value being approached or merely as a process of numerical substitution, rather than as a logical relationship between two interdependent variables.

This difficulty becomes even more pronounced when students encounter the formal definition of limit using ε (epsilon) and δ (delta) notation, which states that for every $\varepsilon > 0$, there exists a $\delta > 0$ such that if $0 < |x - a| < \delta$, then $|f(x) - L| < \varepsilon$ (Fernández, 2004; Boero, 2015; Caglayan, 2015; Adiredja, 2021). This definition requires a high level of abstract, deductive, and relational thinking—capacities that are often not yet fully developed among first-year university students (Kim, Kang, & Lee, 2015; Usman, Hasbi, & Muslem, 2024; Prihandhika & Sopiany, 2025). Despite extensive research documenting students' difficulties with the limit concept, much of the existing literature has primarily focused on identifying misconceptions and procedural errors. Relatively little attention has been given to how students construct the meaning of the formal definition through their ways of thinking. In particular, there is a need to examine how students interpret the ε–δ definition within their evolving cognitive and interpretative frameworks.

According to the Ways of Thinking (WoT) theory proposed by Harel (2008), these difficulties can be understood through the distinction among empirical, procedural, and theoretical ways of thinking. Students operating at the empirical level tend to construct understanding based on direct experience and concrete examples, associating the concept of limit with observable numerical processes (Harel & Sowder, 2013; Koichu, Harel & Manaster, 2013; Blažek & Pech, 2025). At the procedural level, students begin to master computational rules and step-by-step procedures, yet they often lack an understanding of the underlying logical structure. Only at the theoretical level are students able to interpret limit as a relationship between variables that satisfy specific conditions in a deductive manner (Sulastri et al., 2021). In this context, the ε–δ definition is not merely a formal tool, but a manifestation of theoretical thinking that entails an abstract and logically precise understanding of the dependency between two quantities (Gucler, 2014; Fernández-Plaza & Simpson, 2016).

The WoT framework is grounded in the view that mathematical understanding is formed through consistent and reflective patterns of thinking shaped by learning experiences. Within this framework, a distinction is made between Ways of Understanding (WoU) and Ways of Thinking (WoT). WoU refers to the product or form of an individual's understanding of a concept, whereas WoT describes the cognitive and epistemological structures that enable such understanding to emerge (Prihandhika et al., 2020; Sulastri et al., 2022). In other words, WoT represents the underlying mode of reasoning that supports one's way of understanding. This framework classifies ways of thinking into three developmental stages: empirical, procedural, and theoretical (Harel, 2008; Morin et al., 2025). At the empirical stage, students rely on observable, concrete, or numerical experiences. At the procedural stage, they are able to apply algorithms and formal rules without deeply comprehending their underlying meaning. At the theoretical stage, students are capable of connecting concepts to broader logical structures and higher-level abstract principles. In the context of the limit concept, students who think empirically tend to view limit as a "value being approached," those who think procedurally focus on substitution techniques or algebraic manipulation, whereas those who think theoretically understand limit as a functional relationship governed by the formal ε–δ condition (Roh, 2010; Keene, Hall & Duca, 2014; Kidron & Tall, 2015). The development from empirical to theoretical ways of thinking is not linear, but occurs through processes of reflection, cognitive conflict, and epistemological restructuring. Therefore, mathematics instruction should not merely require students to memorize formal definitions, but should also facilitate the negotiation of meaning in a conceptual and reflective manner (Lesh & Harel, 2003).

In the present study, the WoT framework is employed to identify students' patterns of reasoning when they confront the precise definition of limit and the difficulties that arise in understanding the relationship between ε and δ. However, understanding the formal definition of limit is not solely a cognitive matter, but also an interpretative one (Kim & Lim, 2018; Saleh, Budayasa & Lukito, 2025). From this perspective, students' difficulties are not only indicative of incomplete conceptual knowledge but also reflect limitations in how formal mathematical meaning is interpreted and constructed. To make these conceptual distinctions analytically meaningful, the WoT framework was operationalized as a coding scheme for this study. Students' responses from written tasks and interviews were systematically analyzed to identify patterns of reasoning corresponding to empirical, procedural, and theoretical ways of thinking. The classification was guided by specific indicators, including reliance on numerical approximation and direct substitution (empirical), the application of algebraic procedures without explicit conceptual justification (procedural), and the articulation of relational and conditional reasoning involving ε and δ (theoretical).

Furthermore, the distinction between Ways of Understanding (WoU) and Ways of Thinking (WoT) was employed to differentiate between the forms of students' expressed understanding and the underlying reasoning structures that generated them. While WoU was identified through students' final explanations, representations, and conclusions, WoT was inferred from the processes of reasoning, justification patterns, and interpretative strategies evident in their responses. This operationalization enabled a systematic and theoretically grounded analysis of how students construct the meaning of the formal limit definition across different levels of thinking.

The hermeneutic perspective provides a foundation for viewing mathematical understanding as a dialogical process of meaning-making, rooted in the philosophy of interpretation that regards understanding as the outcome of a dialogue between subject and text, and between experience and structures of meaning (Skinner, 1975; Ricoeur, 1978; Ricoeur, 1981; Gallagher, 1992; Gadamer, 2008). In the context of mathematics education, a hermeneutic perspective does not merely emphasize formal logic, but rather focuses on how students interpret the meanings of symbols, definitions, and mathematical representations. From this viewpoint, learning mathematics entails not only mastering symbols and rules, but also interpreting the meanings embedded within mathematical structures. Students encounter mathematical texts that must be interpreted through their prior experiences, linguistic resources, and dialogue with their existing conceptual frameworks (Prihandhika & Perbowo, 2024). Consequently, learning becomes an interaction between preconceptions (prior understanding) and newly encountered formal structures, in which meaning emerges through continuous negotiation and reinterpretation (Carpenter, 2013; Al-Mutawah, 2019; Braithwaite & Sprague, 2021). Thus, students' difficulties in understanding the ε–δ relationship do not simply indicate a deficiency in formal logic, but also reflect limitations in their interpretative processes regarding the meaning conveyed by the definition.

The hermeneutic process involves three essential elements: pre-understanding, interpretative dialogue, and the fusion of horizons (Roberge, 2011). Pre-understanding refers to the initial conceptual schema students bring when encountering new ideas. Interpretative dialogue occurs when students negotiate meaning through interactions with lecturers, peers, or mathematical texts. Fusion of horizons denotes the transformation of understanding that occurs when prior conceptions and new formal structures merge into a deeper and more comprehensive meaning. In calculus instruction, a hermeneutic perspective enables students to perceive the ε–δ definition not as a mere symbolic formality, but as a mathematical language that articulates the idea of precise proximity between function values and points in the domain (Braithwaite & Sprague, 2021). To connect these hermeneutic concepts to the analytical procedure, the present study employed them as interpretative lenses in the coding and analysis of students' responses (Carpenter, 2013). Specifically, pre-understanding was identified through students' initial explanations and intuitive interpretations of limit, particularly those reflecting approximation-based reasoning. Interpretative dialogue was traced in students' attempts to reconcile their prior conceptions with formal definitions, as evidenced in their verbal justifications and shifts in reasoning during interviews. Fusion of horizons was inferred when students demonstrated a transformation in understanding, moving from empirical or procedural reasoning toward a more coherent relational interpretation of the ε–δ definition. These hermeneutic dimensions did not function as separate categories but operated as analytical layers that enriched the interpretation of the WoT-based classifications. In this way, the analysis captures not only the level of students' thinking but also the interpretative processes through which mathematical meaning is constructed .

The integration of the WoT framework and the hermeneutic approach offers a renewed perspective for understanding the dynamics of students' reasoning about the concept of limit. On the one hand, the WoT framework explains the cognitive structures involved in the formation of mathematical thinking by classifying students' reasoning into empirical, procedural, and theoretical categories. However, while this framework effectively captures patterns of reasoning, it does not fully account for how students interpret the meaning of formal definitions or how such meanings are constructed and transformed through learning experiences. The hermeneutic perspective addresses this limitation by focusing on the interpretative processes underlying students' engagement with mathematical concepts. It provides analytical insight into how students' pre-understandings shape their initial interpretations, how meaning is negotiated through interaction with formal definitions, and how shifts in understanding occur through a fusion of horizons. In this sense, the hermeneutic lens does not replace the WoT framework but extends it by revealing the interpretative dimension of mathematical thinking that remains implicit in WoT-based analyses. When combined, these perspectives suggest that the development of students' ways of thinking is not merely the result of algorithmic practice, but rather a hermeneutic process through which meaning is constructed, negotiated, and refined.

Although previous studies on students' understanding of limits have extensively focused on misconceptions, procedural errors, cognitive obstacles, and proof difficulties, limited attention has been given to how students construct meaning when interpreting the formal ε–δ definition as a logical relationship between variables. Existing studies have also rarely connected the Ways of Thinking framework with a hermeneutic perspective to examine how students' meanings evolve from intuitive numerical approximation toward formal structural reasoning. This gap is particularly important in mathematics education because the ε–δ definition requires not only procedural competence but also interpretative understanding of conditional dependency and precision.

Addressing this gap, the present study investigates students' ways of thinking in interpreting the ε–δ definition through a hermeneutic lens. Specifically, the study aims to identify the categories of students' reasoning based on the Ways of Thinking framework and to analyze how mathematical meaning is progressively constructed across empirical, symbolic–procedural, transitional, and structural stages. The novelty of this study lies in integrating the Ways of Thinking theory with hermeneutic interpretation to explain not only the cognitive form of students' reasoning but also the process through which formal mathematical meaning is negotiated. In this way, the study contributes to mathematics education by extending the analysis of the ε–δ definition beyond procedural difficulty toward the interpretative dimension of meaning-making, while also offering focused pedagogical insight for introductory calculus instruction.

Research Methods

This study employed a qualitative approach with a case study design grounded in a hermeneutic perspective, aiming to gain an in-depth understanding of students' interpretative processes in constructing the formal meaning of the limit concept through the interaction among learning experiences, mathematical texts, and conceptual structures (Sharma, 2013; Jahnke, 2014; Bikner-Ahsbahs, Knipping & Presmeg, 2015; Markle, 2021). This approach is situated within a constructivist paradigm that views knowledge as the result of interpretation rather than as the mere accumulation of facts. A case study design was selected because it enables the researcher to explore phenomena contextually and comprehensively within real-life settings (Sharma, 2013). In this study, the case is defined as a single cohort of undergraduate mathematics education students enrolled in a differential calculus course at a public university in West Java, Indonesia. The case is bounded by the instructional context in which the formal ε–δ definition of limit was introduced and discussed, including classroom learning activities, written tasks, and follow-up interviews. Thus, the unit of analysis focuses on students' ways of thinking as they engage with the ε–δ definition within this specific course setting. These boundaries allow the study to capture the situated and interpretative nature of students' reasoning while maintaining a coherent and contextually grounded analysis of how the meaning of limit is constructed.

The research participants were 28 undergraduate students enrolled in a mathematics education program at a university in West Java, Indonesia, who had completed a differential calculus course. All participants were involved in the initial stage of the study through written tasks aimed at exploring their understanding of the limit definition. From this cohort, 12 students were purposively selected for in-depth analysis based on their performance on an initial diagnostic task and supporting classroom observations related to reasoning about the ε–δ definition. Using criteria such as the accuracy of reasoning, use of formal definitions, and coherence of explanations, students were categorized into three levels of conceptual understanding: high, medium, and low. Four students from each category were then selected to ensure balanced representation.

Data were collected through three primary instruments: (1) a written conceptual task on the definition of limit and the application of ε–δ, used to identify general patterns in students' initial ways of understanding and to inform the selection of participants for in-depth analysis; (2) in-depth semi-structured interviews designed to explore students' ways of thinking in interpreting the relationship between ε and δ, as well as their reflections on the formal meaning of limit, which served as the primary source for analyzing reasoning processes; and (3) classroom observations and reflective field notes, employed to capture the dialogical and interpretative dynamics emerging during the learning process and to support the interpretation of students' responses. A total of 12 semi-structured interviews were conducted, one with each selected participant, with each session lasting approximately 25–35 minutes. All interviews were audio-recorded and transcribed verbatim prior to coding and interpretation. The interview protocol was developed based on the WoT framework and directed at identifying empirical, procedural, and theoretical dimensions of students' reasoning. Meanwhile, the observation instrument was informed by hermeneutic principles emphasizing dialogical understanding among students, the lecturer, and mathematical texts (Ricoeur, 1975; Wernet, 2014).

Data analysis was conducted using an interpretative hermeneutic perspective (Gadamer, 1975; Ricoeur, 1981; Brown, 2002; Carl, 2022; Blanco, 2008), emphasizing a cyclical movement between parts and the whole of meaning. The analysis began with open coding of written responses and interview transcripts, in which segments of data were labeled based on students' reasoning patterns related to the ε–δ definition. These initial codes were then grouped into broader categories guided by the Ways of Thinking (WoT) framework, classifying students' reasoning as empirical, procedural, or theoretical based on indicators such as reliance on numerical examples, application of procedures, or use of relational and conditional reasoning. The assignment of categories was carried out iteratively by comparing codes across data sources to ensure consistency. Hermeneutic interpretation proceeded through a continuous movement between individual excerpts and the overall patterns of understanding (whole), allowing the researcher to refine interpretations in light of emerging meanings. This process also involved interpreting students' responses in relation to mathematical texts, prior experiences, and the WoT framework. Finally, thematic synthesis was conducted to identify dominant patterns in students' ways of thinking and their relation to the interpretation of the formal definition of limit.

Trustworthiness was established through triangulation across written tasks, interviews, and classroom observations. In addition, member checking was conducted with selected participants to confirm the alignment between the researcher's interpretations and the reasoning they intended to express. An audit trail of coding and category decisions was also maintained to strengthen transparency and consistency.

All participants provided informed consent prior to data collection. Student identities were anonymized using participant codes (e.g., P-01, P-02), and participation in interviews was voluntary. The study procedures followed institutional ethical guidelines for educational research.

Results and Discussions

This section presents the results of the analysis of written task data from 28 students in a differential calculus course, complemented by in-depth interview data from 12 purposively selected participants. The analysis focuses on students' understanding of the precise definition of limit based on ε (epsilon) and δ (delta). The written data were used to identify general patterns across the cohort, while the interview data provided deeper insights into students' reasoning processes. The analysis was carried out through a thematic coding process guided by the Ways of Thinking framework (Lesh & Harel, 2003; Harel, 2008), and interpreted using a hermeneutic approach to uncover the layers of meaning shaping students' horizons of understanding. The coding indicators used in this analysis are summarized in Table 1.

Table 1. Coding indicators for ways of thinking categories

Ways of Thinking Category	Coding Indicators
Empirical (C1)	Direct substitution, numerical approximation, reliance on graph or table trends, intuitive "approaching value" explanation
Symbolic–Procedural (C2–C3)	Algebraic manipulation, factorization, use of limit laws, procedural symbolic transformation without ε–δ relational explanation
Transitional (C4–C5)	Explicit mention of ε and δ as proximity measures, partial awareness of dependency, incomplete articulation of universal conditional logic
Structural (C6)	Complete explanation of "for every ε there exists a δ," relational dependency between variables, formal logical justification of precision

The findings indicate that students' ways of thinking about limit are not homogeneous, but rather distributed across several categories of conceptual development. Each category signifies a particular mode through which students construct the meaning of limit, progressing from empirically based intuition toward a structurally coherent understanding of the formal ε–δ relationship. These differences reflect not only variations in the level of content mastery, but also the dynamic process of meaning construction that unfolds through interpretative engagement. Overall, the results suggest that most students continue to understand limit within empirical and procedural frameworks, viewing it as an approximate value obtained through substitution or symbolic manipulation. Only a small proportion of students demonstrate a structural understanding of the formal definition of limit as a universal conditional statement that ensures precise proximity between the independent variable and the function value. The distribution of students' ways of thinking across these categories is presented in Table 2.

Table 2. The distribution of students' ways of thinking

Code	Description of Understanding Characteristics	Number of Students	Percentage
C1	Limit is understood as an approximate value obtained through direct substitution or observation of tables/graphs.	12	42.8%
C2–C3	Students apply algebraic manipulation and limit rules without understanding the epsilon–delta relational structure.	9	32.1%
C4–C5	Students begin to mention epsilon and delta but do not yet fully understand their conditional logical relationship.	5	17.8%
C6	Students understand limit as a formal epsilon–delta conditional relationship ensuring functional stability around a point.	2	7.1%
	Total	28	100%

Table 2 indicates that the majority of students (74.9%) remain within the empirical and symbolic–procedural ways of thinking categories. Within these categories, limit is understood as a process of approaching a particular value through direct substitution or algebraic manipulation, without consideration of the formal structure of the ε–δ definition. Approximately 17.8% of students demonstrate a transitional way of thinking, in which they begin to recognize ε and δ as indicators of proximity, yet are not able to logically articulate the conditional relationship expressed as "for every ε there exists a δ." Only 7.1% of students reach the structural ways of thinking category, characterized by understanding limit as a formal relationship between the domain and codomain of a function through ε–δ–based precision constraints.

To ensure analytic transparency, the classification of students into these categories was guided by a coding rubric derived from the ways of thinking framework (Harel & Sowder, 2013). The rubric was developed using explicit indicators observed in students' written responses and interview explanations, focusing on the dominant pattern of reasoning demonstrated across data sources. Empirical classification was assigned when students relied on direct substitution, numerical approximation, or table- and graph-based observations without formal justification. Symbolic–procedural classification was assigned when students successfully employed algebraic manipulation, factorization, or standard limit rules but did not articulate the relational dependency between ε and δ. Transitional classification was used when students explicitly referred to ε and δ as measures of proximity but showed incomplete understanding of the universal conditional relationship. Structural classification was assigned only when students consistently justified the formal statement "for every ε there exists a δ" as a logically constrained relationship between domain and codomain values. These findings reveal the predominance of empirical and procedural ways of thinking in students' understanding of limit, while structural understanding, reflecting the internalization of the formal definition, remains notably limited (Roh, 2010; Boero, 2015). To provide deeper qualitative support for these categorical findings, the following excerpts present English translations of students' interview responses. The translations preserve the original meaning while ensuring academic clarity. To illustrate the variation in students' reasoning across the purposively selected high-, medium-, and low-level participants, representative excerpts from all 12 interviewees are presented below.

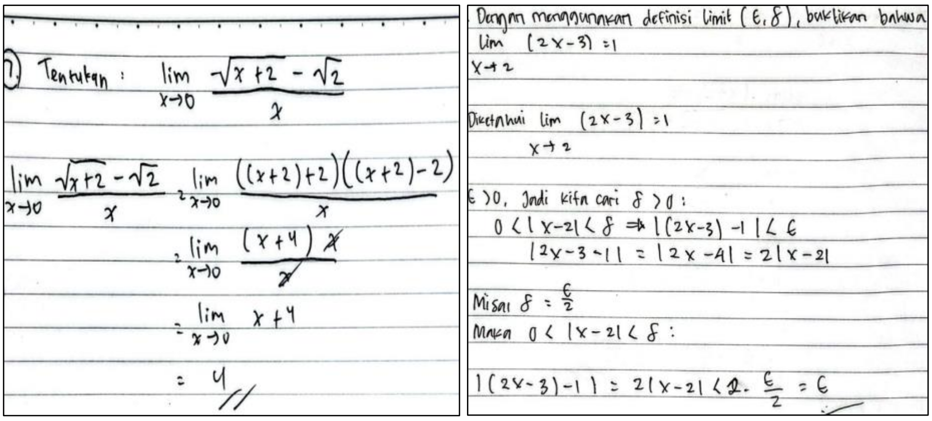
In the Empirical Ways of Thinking (C1) category, students tended to interpret limit as numerical approximation: "A limit means that when x approaches a certain number, the function value also gets closer to that number. So I usually just substitute the number directly" (P-02). "From the table, if the values keep getting closer to 4, then the limit must be 4" (P-11). "I usually look at the numbers around the point first, because if they are close enough, the value of the limit can already be identified" (P-03).

These responses reflect reliance on perceptual trends and substitution-based reasoning. In the Symbolic–Procedural Ways of Thinking (C2–C3) category, students emphasized algebraic manipulation: *"I use limit rules or factor the expression first so that I can substitute the value" (P-05). "As long as the expression can be simplified, the result will come out" (P-08). "The important thing is to transform the expression into a form that can be directly substituted" (P-06).*

Here, symbols function primarily as computational tools rather than as representations of logical relationships. In the Transitional Ways of Thinking (C4–C5) category, students began recognizing epsilon and delta but struggled with their logical structure: *"Epsilon represents the distance of the result, and delta represents the distance of x, but I'm still confused about why it has to be for every epsilon" (P-14). "I know there is a relationship between epsilon and delta, but I'm not fully sure how to prove it" (P-17). "Delta seems to depend on epsilon, but I still do not understand how the relationship is always guaranteed" (P-16).*

These excerpts indicate emerging but incomplete structural awareness. Finally, in the Structural Ways of Thinking (C6**)** category, students demonstrated integrated reasoning: *"For every epsilon we choose, there must be a delta so that the function's value stays within that bound. So the limit is about maintaining the function's stability around a point" (P-23). "The epsilon–delta definition ensures that the approximation is truly precise,* *not just based on the graph" (P-27). "The meaning of the limit is not only approaching, but proving that the distance of the function value can always be controlled through delta" (P-25).*

These responses reveal a coherent understanding of the formal relational structure underlying the definition of a limit. Taken together, the categorical distribution and the representative excerpts demonstrate that students' understanding of the limit concept is not uniformly structured but layered across distinct interpretative horizons. The predominance of empirical and symbolic–procedural orientations indicates that most students operate within approximation-based or algorithmic frameworks, where meaning is constructed through substitution, pattern recognition, or algebraic manipulation. Only a limited number of students exhibit structural reasoning characterized by relational precision and logical conditioning. To further illustrate the predominance of procedural reasoning identified in this study, Picture 1 presents a representative example of a student's written response categorized within the empirical and symbolic–procedural orientation (Prihandhika & Sopiany, 2025).



Picture 1. Empirical and Symbolic–Procedural orientation

The selected work exemplifies how students rely heavily on algebraic manipulation and formal limit rules while remaining detached from the underlying relational structure of the epsilon–delta definition. In this response, the student successfully applies factorization and substitution techniques to obtain the correct numerical result. However, the reasoning process is confined to symbolic transformation without any explicit articulation of dependency relationships between variables or acknowledgment of universal logical conditions (Kidron, 2015).

The focus remains on transforming the expression into a solvable form, rather than on justifying why the limit exists in terms of bounded relational precision. As seen in Picture 1, the student's solution progresses through systematic algebraic steps: rewriting the expression, factorizing, canceling common terms, and performing direct substitution. Each transformation is procedurally valid and leads to the correct numerical conclusion. Nevertheless, the solution does not reference the formal definition of limit, nor does it articulate any relational justification involving epsilon and delta. The absence of explicit conditional reasoning suggests that the student conceptualizes limit as the endpoint of symbolic simplification rather than as a logically constrained relationship (Kidron, 2015; Kim, 2015). The argument is completed once the expression becomes computable, indicating that procedural closure substitutes for conceptual justification (Gucler, 2014). In this sense, correctness is achieved operationally, but not structurally. This example therefore supports the broader finding that procedural dominance can mask underlying conceptual gaps. Students can successfully execute algorithmic transformations while remaining epistemologically distant from the formal meaning of the limit definition. The visual evidence provided in Picture 1 thus substantiates the interpretative claim that most students operate within symbolic–procedural horizons rather than structural ones.

This stratification suggests that the epsilon–delta definition has not yet been fully internalized as a formal conceptual structure. Instead, students' reasoning appears to evolve through gradual reinterpretation—from perceptual and procedural engagement toward structural integration. These findings raise important theoretical questions regarding how mathematical meaning is constructed, stabilized, and transformed within the learning process. The following discussion situates these results within the *Ways of Thinking* framework and explores their implications for conceptual development in calculus learning (Abdul-Wasiu et al., 2024).

The findings of this study suggest that students' construction of meaning regarding limit develops progressively across distinct ways of thinking. Within the Ways of Thinking framework (Harel, 2008; Harel & Sowder, 2013), the predominance of empirical and symbolic–procedural categories indicates that most students continue to understand limit as a numerical or procedural phenomenon rather than as a logical structure conditionally relating ε and δ. This interpretation is directly supported by the interview excerpts, in which students frequently described limit as a value obtained through substitution, approximation, or symbolic simplification.

At the empirical ways of thinking stage, the data show that students tend to interpret limit as a value that approaches a particular number through direct substitution or numerical observation. These responses reflect reliance on concrete experience and perceptual reasoning. This pattern may be interpreted as evidence that students have not yet distinguished between the value of a function and the value of its limit at a given point. In the Symbolic–Procedural stage, students demonstrate increased use of symbols and formal rules, as shown in their emphasis on factorization, algebraic transformation, and standard limit laws. However, the presented data also indicate that these symbols are primarily treated as computational tools, with limited explicit reference to the formal ε–δ dependency. In the Transitional category, the interview excerpts provide evidence that students begin to recognize ε and δ as measures of proximity, yet remain uncertain about the universal conditional structure expressed in the formal definition. This suggests an emerging awareness of relational precision, although the logical dependency between variables is not yet fully articulated. From a hermeneutic perspective, this stage may be interpreted as a moment in which prior approximation-based meanings are being renegotiated in relation to formal mathematical language (Markle, 2021; Carl, 2022).

The structural ways of thinking category is directly evidenced by students' explicit statements that "for every ε there exists a δ" ensuring controlled function behavior around a point. These excerpts support the claim that some students have reached conceptual integration of the formal definition (Fernández, 2004).

From a hermeneutic lens, this more coherent relational reasoning may be interpreted as a convergence between students' prior intuitive understanding and the formal logical structure of the ε–δ definition (Blanco, 2008). Overall, the data support the conclusion that understanding the precise definition of limit is unlikely to develop through procedural practice alone. Rather, the findings indicate the importance of instructional approaches that create opportunities for conceptual reflection, explicit discussion of variable dependency, and interpretative dialogue connecting intuitive approximation with formal conditional reasoning. Accordingly, calculus instruction should be directed toward cultivating awareness of mathematics as a system of logical relationships rather than merely a collection of symbolic procedures.

Conclusions and Suggestions

This study found that most students' understanding of the ε–δ definition of limit remained within empirical and symbolic–procedural ways of thinking, with only a small proportion demonstrating structural understanding. The findings indicate that students commonly rely on numerical substitution, algebraic simplification, and procedural limit rules, while only a few are able to articulate the formal conditional dependency between ε and δ.

From a theoretical perspective, these results suggest that students' understanding of the formal limit definition develops progressively from approximation-based reasoning toward relational and conditional reasoning. Within the scope of this study, the hermeneutic perspective helps explain how students negotiate meaning between prior intuitive experiences and formal mathematical structures, particularly in moving from substitution-based interpretations toward explicit awareness of variable dependency.

In practical terms, the findings suggest the importance of instructional activities that explicitly connect intuitive approximation with the formal ε–δ relationship. For example, calculus instruction may include guided tasks requiring students to explain why a selected δ satisfies a given ε condition, classroom discussions focused on conditional dependency, and reflective comparisons between substitution-based and formal reasoning. These suggestions are limited to the context of introductory differential calculus and should be interpreted as focused implications rather than broad generalizations.

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