

Exploring the Growth of a Junior High School Student's Mathematical Understanding and Higher-Order Thinking

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Abstract:

Mathematical understanding develops dynamically and plays an important role in fostering higher-order thinking skills (HOTS). However, studies using the Pirie–Kieren framework have not consistently examined the relationship between the growth of mathematical understanding and higher-order thinking. This study aims to explore the growth of a junior high school student's mathematical understanding at Junior High School Number 8 Lubuklinggau, and higher-order thinking within the framework of Pirie-Kieren. This study employed a descriptive exploratory approach involving one eighth-grade student with high mathematical ability, selected purposively. The student completed a System of Linear Equations in Two Variables (SPLDV) task designed based on the Pirie–Kieren layers of understanding. Data were collected through observations, interviews, and documentation, and were analyzed thematically. The results show that the student's mathematical understanding exhibited growth in a dynamic and nonlinear manner, progressing from primitive knowing to structuring, with recurring folding-back. Along this progression, the student demonstrated higher-order thinking, including recognizing concepts, applying prior knowledge in new contexts, constructing mathematical models, identifying properties, abstracting ideas, reflecting on processes, and connecting concepts. These findings indicate that the growth of mathematical understanding is closely associated with the development of higher-order thinking, suggesting that both evolve simultaneously during the learning process.

Keywords: mathematical understanding growth; Pirie–Kieren theory; higher-order thinking skills; folding back; SPLDV

Introduction

Mathematical understanding is a fundamental ability that students must master and a key concern in developing effective mathematics learning (Carpenter & Lehrer, 1999; Gulkilik et al., 2020). In recent decades, this concept has been increasingly explored through the Pirie–Kieren theoretical framework, which views understanding as a dynamic and non-linear process consisting of interconnected layers (Pirie & Martin, 2000). This framework helps explain how students build, organize, and reorganize their knowledge over time. Research shows that this growth involves a



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mechanism known as folding back, where students return to earlier levels/layers to strengthen their understanding (Efuansyah et al., 2024a).

With appropriate scaffolding, students can then progress to higher levels, such as from image having to property noticing, which supports deeper conceptual mastery (Efuansyah et al., 2024b; Martin & Towers, 2015). This strong conceptual understanding becomes an important foundation for engaging in Higher-Order Thinking Skills (HOTS), which involve the ability to connect, manipulate, and transform knowledge critically and creatively to solve problems in new situations (Chai & Kong, 2017; Lewis & Smith, 1993; Thomas & Thorne, 2009). In the 21st century, HOTS are essential for helping students adapt to increasingly complex challenges (Chai & Kong, 2017; Larsson, 2017), as they require processes such as critical thinking, decision-making, and integrating new information with prior knowledge (Miterianifa et al., 2021).

Various previous studies have examined the growth of mathematical understanding using the Pirie–Kieren framework, both in individual and collaborative contexts. Martin & Towers (2009) investigated the growth of collective understanding through improvisational coactions, while Codes et al. (2013) described students' understanding of infinite numerical series. Gokalp & Bulut (2018) explored multiple representations within the Pirie–Kieren model, and Sengul & Yıldız (2016) examined multivariable functions using this theoretical lens. Patmaniar et al. (2021) examined folding-back processes in problem-solving by gender, while Zawawi et al. (2023) analyzed students' mathematical understanding through the Pirie–Kieren lens. Additionally, Chuene et al. (2023) investigated how classroom talk supports learners' folding back into their understanding of geometry.

However, these studies have generally not explicitly linked the growth of mathematical understanding with the emergence of higher-order thinking skills. Although previous studies have explored the growth of students' mathematical understanding using the Pirie–Kieren framework, they have not explicitly examined how this growth relates to the emergence of higher-order thinking skills (HOTS). For instance, Al-Wardat et al. (2026) employed the Pirie–Kieren model primarily to describe students' progression of understanding and to assess its impact on specific cognitive outcomes, such as geometric thinking. However, their study did not examine how this understanding connects to the development of higher-order thinking skills. As a result, there is limited understanding of how a junior high school student's mathematical understanding and higher-order thinking evolve together throughout the learning process. This gap is important, as mathematics education in the 21st century requires not only conceptual understanding but also the ability to think critically, reflectively, and creatively. Therefore, it is necessary to explore the growth of a junior high school student's mathematical understanding and higher-order thinking in an integrated manner.

The novelty of this study lies in examining the integration of growth in mathematical understanding with higher-order thinking within the Pirie–Kieren framework. It provides a comprehensive view of how a junior high school student's layered understanding develops alongside higher-order thinking, showing that both aspects evolve simultaneously rather than separately. This study also offers practical insights for mathematics educators, especially in designing learning experiences that support both conceptual understanding and higher-order thinking by attending to students' folding-back processes. From a theoretical perspective, it extends the Pirie–Kieren framework by identifying characteristics at each level of understanding associated with higher-order thinking skills. Therefore, this study aims to explore the growth of a junior high school student's mathematical understanding and higher-order thinking using the Pirie–Kieren theory. Specifically, it focuses on how the student progresses through understanding layers when solving System of Linear Equations in Two Variables (SPLDV) tasks and how the folding-back mechanism contributes to the development of critical, reflective, and creative thinking skills.

Research Methods

This study employed a descriptive-exploratory design to investigate in depth the growth of a junior high school student's mathematical understanding and its relation to higher-order thinking, particularly through the lens of the Pirie–Kieren theoretical framework. Such a design is appropriate for examining phenomena in which variables and conceptual relationships are not yet fully defined, thereby enabling a detailed exploration of meaning-making processes (Creswell, 2014). The subject of the study was an eighth-grade student from SMPN 8 Lubuklinggau, purposively selected based on demonstrated high ability in solving tasks on the System of Linear Equations in Two Variables (SPLDV). The focus on a single subject was intentional, as the study sought to capture the dynamics of cognitive processes rather than to generalize findings. Understanding how mathematical understanding develops alongside higher-order thinking requires close, continuous examination of an individual's reasoning, best achieved through an intensive case study approach.

The identification of the subject, coded as CL, was carried out in consultation with a mathematics teacher to ensure the participant met the required level of mathematical proficiency. The research instrument consisted of an SPLDV task that was systematically developed through several stages, beginning with a content analysis of the topic and its alignment with the levels of understanding in the Pirie–Kieren framework, namely primitive knowing, image making, image having, property noticing, formalising, observing, and structuring. Based on this alignment, the instrument was constructed and refined into eight sub-items, each explicitly designed to represent a particular level of understanding while capturing the progression of conceptual growth. To ensure its quality, the instrument underwent a two-stage validation process involving expert judgment by two professors of mathematics

education, who evaluated its content relevance, clarity, and theoretical alignment. Their feedback informed revisions to wording and difficulty level, and the revised instrument was subsequently re-examined to confirm its consistency and appropriateness, resulting in adequate content validity and strong alignment with the theoretical framework.

Data collection was conducted through observation, task-based interviews, and documentation, using clearly structured, interrelated procedures to comprehensively capture the subject's thinking processes. During the observation phase, the subject was individually observed while working on the SPLDV tasks in a controlled setting, with all activities recorded using video and detailed field notes to capture both verbal and non-verbal responses, particularly those indicating movement across the Pirie–Kieren levels of understanding. This phase was followed by task-based interviews conducted using a semi-structured protocol, consisting of guiding questions aligned with each level of understanding while allowing flexibility for probing based on the subject's responses. The interviews were designed to progressively explore the subject's thinking, beginning with initial understanding and extending toward deeper conceptual reasoning. When the subject encountered difficulties at a particular level, the researcher applied structured prompting strategies, such as encouraging the subject to revisit prior reasoning or reconsider earlier representations, in line with the folding back mechanism to facilitate the reconstruction and strengthening of understanding.

All interviews were audio-recorded and transcribed verbatim, and the collected data, including written work, observation notes, and transcripts, were systematically organized for analysis. Data analysis was carried out through several stages of coding and interpretation guided by the Pirie–Kieren framework, beginning with open coding to label data using predefined codes for participants and levels of understanding, followed by axial coding to identify relationships, transitions, and patterns of growth, including instances of folding back, and concluding with selective coding to synthesize themes describing the trajectory of the subject's mathematical understanding and its contribution to higher-order thinking. Data interpretation was consistently aligned with the theoretical framework, ensuring coherence and depth of analysis. To enhance the credibility of the findings, validation techniques such as triangulation and member checking were employed, in which data from multiple sources were cross-verified, and interpretations were confirmed with the subject. Through these systematic procedures, the study ensured the rigor and trustworthiness of its qualitative findings while providing a detailed account of the growth of mathematical understanding and higher-order thinking.

Results and Discussions

Based on an interview with the eighth-grade mathematics teacher, one female student was purposively selected as a candidate subject due to her high mathematical

ability, particularly in solving SPLDV tasks. This selection was intended to support an in-depth exploration of the growth of mathematical understanding and its relationship with higher-order thinking within the Pirie–Kieren framework. The chosen subject, coded as CL, was considered capable of demonstrating rich cognitive processes that could reveal how understanding develops across different levels. The following section presents excerpts from interviews with CL, conducted to deepen understanding of the progression of her mathematical understanding and its contribution to the development of higher-order thinking skills.

The Process of CL's Mathematical Understanding Growth and Higher-Order Thinking

Primitive Knowing Level of CL

CL understood the given problem and provided detailed information, including the known aspects of the problem and general questions. Excerpts from the interview with CL at the primitive knowing level are as follows.

P : Try to understand part a first. What situation do you see, CL?

CL : One pair of pants plus two glasses equals one hundred thousand.
Three pairs of pants plus one glass equals one hundred thousand.

P : How do you write it?

CL : Alright, sir, here it is:

1 celana + 2 kacamata = Rp. 100.000
3 celana + 1 kacamata = Rp. 100.000

1 pant + 2 glasses = 100.000
3 pants + 1 glasses = 100.000

CL's performance at the primitive knowing level serves as an essential starting point for the growth of mathematical understanding, as conceptualized in the Pirie–Kieren framework. At this level, CL can read and comprehend the problem, identify relevant information from visual representations, and establish initial connections, such as relating the number of pants and glasses to a total of one hundred thousand rupiah. However, her representation of this understanding remains limited to simple and informal mathematical expressions, indicating that her reasoning is still closely tied to the context and has not yet progressed toward abstraction. This finding signifies that CL is operating within a stage where understanding is constructed through recognition rather than formal reasoning, which aligns with previous studies suggesting that learners begin by identifying key relationships before advancing through processes such as folding back to achieve deeper understanding (Hähkiöniemi et al., 2022; Patmaniar et al., 2021).

In relation to Higher-Order Thinking Skills (HOTS), this stage demonstrates that higher-order thinking does not emerge instantly but develops progressively from foundational understanding, as CL's ability to recognize relationships reflects the early formation of cognitive processes that support analysis and evaluation. This indicates

a strong theoretical connection between the growth of mathematical understanding and HOTS, where each level of understanding contributes to increasingly complex reasoning. From an educational perspective, this finding implies that mathematics instruction should support students at this initial stage through meaningful contexts, guided questioning, and opportunities to articulate ideas, thereby facilitating the transition to higher levels of understanding. Therefore, primitive knowing should not be viewed as a limitation, but as a critical entry point that underpins the development of more advanced mathematical thinking and higher-order cognitive skills.

Image Making Level of CL

Excerpts from the interview with CL regarding the Image Making level.

P : Let's go back to question point b. Do you know what 'system' means?

CL : Not yet, sir.

P : $3x = 6000$, this is an equation. Now, a system of equations is a collection or combination of two or more equations. Okay, let's go back to question point b. Create a mathematical model from the situation that was written!

CL : Yes, like this, sir.

$$\begin{aligned} 1x + 2y &= 100.000 \\ 3x + 1y &= 100.000 \end{aligned}$$

In understanding the material on Systems of Linear Equations in Two Variables (SPLDV), CL initially experienced difficulties when focusing on a single equation, indicating that her understanding had not yet supported integrated and relational mathematical reasoning. This difficulty reflects a conceptual limitation in coordinating multiple representations simultaneously, which is essential in solving SPLDV problems. Through the use of prompting questions, CL engaged in a folding-back process, returning to more basic knowledge to reconstruct and deepen her understanding (Widyastuti et al., 2023). This finding suggests that the growth of mathematical understanding, as described in the Pirie–Kieren framework, is inherently nonlinear and recursive, in which progress often requires revisiting earlier levels to strengthen conceptual foundations. In relation to Higher-Order Thinking Skills (HOTS), this process demonstrates that higher-order thinking develops through the flexible reorganization of prior knowledge rather than emerging instantly from complex tasks.

As emphasized by Dewi and Machromah, creative thinking in HOTS involves flexibility in using existing knowledge and generating new solution strategies (Dewi & Machromah, 2022), as evidenced in CL's ability to adjust her reasoning after revisiting fundamental concepts. Furthermore, CL's ability to distinguish between Linear Equations in Two Variables (PLDV) and Systems of Linear Equations in Two Variables (SPLDV), as well as to apply prior knowledge in a more structured way, indicates a transition toward higher levels of understanding, where more systematic

internal representations begin to form (Rusadi et al., 2022; Widyastuti et al., 2023). This finding highlights a strong theoretical connection between the growth of mathematical understanding and HOTS, where increasingly organized conceptual structures support analytical and evaluative thinking. From an educational perspective, this implies that mathematics instruction should emphasize not only procedural accuracy but also opportunities for students to revisit, reflect, and reconstruct their understanding through guided questioning. Therefore, these findings indicate that the development of higher-order thinking is closely linked to the dynamic, iterative growth of mathematical understanding, reinforcing the importance of instructional practices that align with students' cognitive development.

Image Having Level of CL

Excerpts from the interview conducted with CL at the Image Having level.

P : That is what is called equivalent. Why is that?

CL : Why is that, sir...

P : Because the value being substituted is the same. In $2p$ and $3p$, the value of p is also 500. Another example: if $2p = 4000$, $3p = 6000$. What about 9000 for $4p$? Is that equation equivalent?

CL : No, sir, because if $2p = 4000$ and $3p = 6000$, that means $p = 2000$. If $p = 2000$, then $4p$ should be 8000.

P : Great, let's go back to task point c1. Based on point b, both sides are multiplied by 2. Is the resulting equation equivalent to the equation in point b?

CL : If I substitute, for example, $x = 20000$ and $y = 40000$ into point b, then both equations become true. If I substitute the same x and y into the equations in point c1, then both equations are also true. That means the equations formed in c1 are equivalent to the equations in point b, sir.

$$\begin{array}{l} 2x + 4y = 200.000 \\ 6x + 2y = 200.000 \end{array} \left. \vphantom{\begin{array}{l} 2x + 4y = 200.000 \\ 6x + 2y = 200.000 \end{array}} \right\} \text{equivalen} \begin{array}{l} 1x + 2y = 100.000 \\ 3x + 1y = 100.000 \end{array}$$

P : Yes, that's correct, good analysis. What about the equations formed in c2, where both equations from point b are added by 20000? Are the resulting equations equivalent to the equations in point b?

CL : So the equations formed would be like this:

$$\begin{array}{l} 1x + 2y + 20.000 = 120.000 \\ 3x + 1y + 20.000 = 120.000 \end{array} \left. \vphantom{\begin{array}{l} 1x + 2y + 20.000 = 120.000 \\ 3x + 1y + 20.000 = 120.000 \end{array}} \right\} \text{equivalen} \begin{array}{l} 1x + 2y = 100.000 \\ 3x + 1y = 100.000 \end{array}$$

P : Yes.

CL :

They are also equivalent, sir, because if I substitute the values $x = 20000$ and $y = 40000$, both equations are true. This is the same as the equations in point b, which are also true when I substitute those values.

When faced with the task of determining other equivalent equations, CL initially experienced confusion and was unable to independently construct alternative forms, indicating a limitation in conceptual flexibility rather than merely procedural knowledge. This difficulty suggests that her understanding had not yet developed to the point of supporting transformation and generalization, which are essential aspects of higher-order thinking. However, through prompting questions, CL engaged in a folding-back process, returning to the image-making level, where she recalled that equivalent equations can be formed by applying the same operations to both sides.

This process enabled her to reconstruct her understanding and recognize that different but equivalent mathematical models could be generated without changing the meaning of the equations (Deciku et al., 2022; Nurzannah et al., 2021). As a result, CL demonstrated a shift to the image having level, where her understanding became more stable and could be applied consistently in generating equivalent forms. This finding aligns with the Pirie–Kieren theory, which emphasizes that the growth of mathematical understanding occurs through recursive processes such as folding back that reorganize and strengthen prior knowledge (Rahmadiani et al., 2024). In relation to Higher-Order Thinking Skills (HOTS), CL's progression reflects the development of analytical and flexible thinking, as she moved from relying on surface-level recognition to understanding underlying mathematical structures.

This supports the view that HOTS emerge through the refinement and integration of conceptual understanding rather than through isolated exposure to complex problems. Furthermore, learning in the context of SPLDV becomes more meaningful when students are supported in building connections among concepts, thereby structuring their understanding and making it more transferable (Deciku et al., 2022; Talenti, 2011). Consistent with Asih (2018), this process also illustrates how critical thinking skills are embedded in problem-solving activities that require students to actively construct and adapt their knowledge. From an educational perspective, these findings suggest that mathematics instruction should emphasize guided questioning and opportunities to explore multiple representations, as such approaches facilitate the folding back process and promote the development of higher-order thinking. Therefore, this result highlights that the growth of mathematical understanding and higher-order thinking is an interconnected, dynamic process central to this study's focus on students' cognitive development.

Property Noticing Level of CL

Excerpts from the interview with CL regarding the Property Noticing level are as follows.

P : How will it be? Try starting from the beginning.

CL : $2a$ plus $3b$ equals 13 , multiplied by 1 to keep the value, so $2a$ plus $3b$ equals 13 .

a plus $2b$ equals 8 , multiplied by 2 , so $2a$ plus $4b$ equals 16 .

P : Yes, perhaps like this. Let me help write it down so it's easier to solve:

$$\begin{array}{l} 2a + 3b = 13 \quad | \times 1 \longrightarrow 2a + 3b = 13 \\ a + 2b = 8 \quad | \times 2 \longrightarrow 2a + 4b = 16 \end{array}$$

Now, how do you eliminate variable a ?

CL : Subtract them, sir. So it becomes $-b = -3$, therefore $b = 3$.

P : Good, so it becomes like this:

$$\begin{array}{r} 2a + 3b = 13 \quad | \times 1 \longrightarrow 2a + 3b = 13 \\ a + 2b = 8 \quad | \times 2 \longrightarrow 2a + 4b = 16 \quad - \\ \hline -b = -3 \\ b = 3 \end{array}$$

CL : Yes sir. That seems to be the way it is.

P : Yes, that is called the elimination method. Now, after obtaining the value of b , how do we get the value of a ?

CL : Um... I'm confused, sir.

P : Another example:

$$\begin{array}{l} 1 \text{ bread} + 2 \text{ candies} = 2000, 1 \text{ bread is } 1000 \text{ means:} \\ 1000 + 2 \text{ candies} = 1000 \\ 2 \text{ candies} = 1000 \\ 1 \text{ candy} = 1000 : 2 \\ 1 \text{ candy} = 500 \end{array}$$

That's better, what is that method called?

CL : I don't know, sir.

P : That is called the substitution method. So, you can solve them using only the elimination method, only the substitution method, or both (the mixed method). Good, if you already understood, let's go back. How do we obtain one of the variables, x or y , from the following equations?

CL : Alright, sir.

$$\begin{array}{l} x + 2y = 100.000 \\ 3x + y = 100.000 \end{array}$$

I want to eliminate x , so the top equation is multiplied by 3 and the bottom equation is multiplied by 1 .

P : Try working on it. Write it down. How would you do it?

CL : Alright, sir.

$$\begin{array}{l} x + 2y = 100.000 \quad | \times 3 \\ 3x + y = 100.000 \quad | \times 1 \end{array} \left| \begin{array}{l} 3x + 6y = 300.000 \\ 3x + y = 100.000 \quad - \\ \hline 5y = 200.000 \\ y = 40.000 \end{array} \right.$$

Is it like this, sir?

P : $x + 2y = 100.000$
 $x + 2 \times 10.000 = 100.000$
 $x + 20.000 = 100.000$
 $x = 20.000$

Good.

CL demonstrated a notable ability to apply mixed methods, namely elimination and substitution, in solving problems involving systems of linear equations in two variables (SPLDV), which reflects a meaningful development in her mathematical understanding and higher-order thinking. Although she initially experienced difficulty with the substitution method, this challenge can be interpreted as a gap in connecting procedural knowledge with its underlying conceptual meaning. Through guided questioning, the researcher facilitated a folding back process that led CL to revisit the fundamental concept of substitution, enabling her to reconstruct her understanding and apply it more effectively. This finding suggests that conceptual difficulties often arise not from a lack of knowledge but from an incomplete integration of prior knowledge, which can be strengthened through targeted instructional support.

Such a process aligns with studies indicating that students learn more effectively when they are guided to recall and reorganize basic mathematical concepts, thereby deepening their understanding (Awantagusnik & Prastyo, 2024; Anggraini et al., 2023). As CL began to recognize that the problem could be approached through substitution, she demonstrated increased cognitive flexibility, a key characteristic of Higher-Order Thinking Skills (HOTS). This progression indicates that HOTS are not merely reflected in solving complex problems, but in the ability to select, adapt, and connect multiple strategies based on conceptual understanding. Furthermore, CL's ability to employ different solution methods signifies her transition to the property noticing level within the Pirie–Kieren framework, where students begin to understand relationships between mathematical concepts rather than simply applying procedures. This stage is crucial because it marks the emergence of analytical thinking as students begin to recognize the patterns and structures underlying mathematical operations. Consistent with Martin (2008), a strong conceptual understanding supports the ability to solve more complex problems and is closely related to the folding back process, which allows students to revisit and refine their knowledge to achieve better solutions (Anggraini et al., 2023).

Additionally, engaging in such deep learning processes encourages students to explore alternative strategies and construct more flexible problem-solving approaches (Setiyani et al., 2021; Tani et al., 2024). From an educational perspective, these findings highlight the importance of instructional practices that promote conceptual understanding through guided inquiry, multiple solution methods, and reflective thinking. Therefore, this result indicates that the growth of mathematical understanding and higher-order thinking is interconnected and dynamic, in which the

ability to integrate and flexibly apply knowledge is a key indicator of meaningful mathematical learning, as emphasized in this study exploring students' cognitive development.

Formalizing Level of CL

The results of the interview with CL at the Formalizing level are as follows.

P : -- = -, +- = +, Try this, what is the result if, for example, this is $3 - (-3) = \dots$
 $3 + (-3) = \dots ?$

CL : 0 and 6, sir?

P : Try to understand again. Is that correct?

CL : Oh, it's wrong, sir. It should be $3 + 3 = 6$ and $3 - 3 = 0$.

P : So, try to correct it. ++ = ..., -- = ..., +- = ..., -+ = ...
 $++ = +$, $-- = +$, $+- = -$, $-+ = -$

Good. Now, let's go back to the previous solution:

$$\begin{array}{r|l} 2x + 3y = 3 & \times 3 \\ 3x - y = 10 & \times 2 \end{array} \quad \begin{array}{r} 6x + 9y = 9 \\ 6x - 2y = 20 \end{array} \quad \begin{array}{r} - \\ \hline 11y = -11 \\ y = -1 \end{array}$$

CL : Is it like this, sir?

P : Yes, good. Now continue to find the value of x!

CL :

$$\begin{array}{l} 2x + 3y = 3 \\ 2x + 3 \cdot (-1) = 3 \\ 2x + (-3) = 3 \\ 2x - 3 = 3 \\ 2x = 3 + 3 \\ 2x = 6 \end{array}$$

P : So, what is the value of x?

CL : $2x = 6$, so $x = 6 : 2$, $x = 3$.

P : Okey

In the context of this study on Exploring the Growth of a Junior High School Student's Mathematical Understanding and Higher-Order Thinking, CL's ability to solve systems of linear equations in two variables (SPLDV) using the elimination method by equating coefficients reflects not only procedural competence but also a deeper development of conceptual understanding. This finding indicates that the elimination method serves as a cognitive tool that helps students recognize structural relationships within SPLDV, thereby facilitating movement toward more formal reasoning. Within the Pirie-Kieren framework, this progression can be interpreted as CL's transition to the formalising level, where understanding is no longer tied to specific contexts but begins to take the form of generalized and consistent procedures.

Previous studies have shown that the Pirie–Kieren model effectively supports such development across levels of understanding, particularly in enabling students to construct formal mathematical processes (Sengul & Yıldız, 2016; Zawawi et al., 2023). However, CL's experience with errors in numerical operations reveals that reaching a higher level of understanding does not eliminate the possibility of misconceptions; rather, it highlights the dynamic and recursive nature of learning. The intervention that required CL to revisit the properties of operations illustrates the critical role of folding back in correcting errors and reinforcing conceptual clarity (Martin & Towers, 2016; Somasundram et al., 2019). This process explains why CL ultimately refined her solution strategy: revisiting foundational knowledge enabled her to reorganize and stabilize her understanding. From the perspective of Higher-Order Thinking Skills (HOTS), CL's ability to abstract solution steps into a more systematic procedure signifies the emergence of algorithmic thinking, which involves analysis, generalization, and the ability to apply knowledge across different situations (Tanujaya et al., 2020).

This demonstrates that HOTS are not separate from conceptual understanding, but rather develop through the refinement and structuring of that understanding. Furthermore, integrating practical problem-solving strategies with theoretical knowledge, as demonstrated by CL, is essential for fostering critical and analytical thinking in mathematics (Greco, 2023; Patmaniar et al., 2021). From an educational perspective, these findings imply that mathematics instruction should emphasize not only procedural fluency but also reflective learning processes, such as revisiting prior knowledge, analyzing errors, and encouraging students to generalize solution strategies. Therefore, this result indicates that the growth of mathematical understanding and higher-order thinking is a dynamic, interconnected process, in which the ability to formalize, reflect, and generalize knowledge becomes central to meaningful mathematical learning.

Observing Level of CL

The following are the results of the interview with CL.

P : How did you do the elimination?

CL : First, I looked at the coefficients of the x and y variables from both equations, which one was easier to solve. For example, I chose x . Then I made the coefficients of x the same in both equations by cross-multiplying the two coefficients of x . After forming two new equations, I operated on them (subtraction) which finally resulted in the value $y = -1$.

Pertama mengeliminasi variabel x sehingga ketemu angka dari variabel y , lalu mensubstitusi agar ketemu angka dari variabel x

- P** :

First, eliminate variable X to obtain the value of variable Y, then substitute it to find the value of variable X.
--
- CL** : Okay, what about the substitution method?
- P** : After I obtained the value $y = -1$, I substituted the value $y = -1$ into the equation $2x + 3y = 3$, and finally obtained $x = 3$.
- CL** : Does it have to be the equation $2x + 3y = 3$?
- P** : No, sir, it could also be $3x - y = 10$. It's just that I chose the equation $2x + 3y = 3$ because it is easier to operate with.

Nice...

CL's solution process for the SPLDV problem demonstrates significant development in both mathematical understanding and higher-order thinking, as explored in this study. She began by eliminating one variable through equalizing coefficients, carefully observing the coefficients of x and y and selecting the variable that was more efficient to eliminate. This choice indicates not only procedural knowledge but also strategic decision-making, which reflects an emerging level of analytical thinking. CL then multiplied each equation to align the coefficients, subtracted to obtain y , and then substituted this value into one of the equations to find x . Importantly, CL recognized that substitution could be applied to either equation but deliberately chose the simpler one, demonstrating flexibility and efficiency in her reasoning. This process suggests that her thinking had moved beyond routine procedures toward a more reflective and strategic approach.

From the perspective of the Pirie–Kieren framework, this indicates that CL was operating at a formalising level, where previously constructed knowledge is coordinated into a structured and consistent problem-solving scheme. The fact that CL revisited the concept of elimination—specifically the rationale of equalizing coefficients before subtraction—illustrates the role of internal reflection in strengthening conceptual understanding. This reflective activity explains why CL was able to apply the method consistently and accurately, as her understanding was not merely procedural but conceptually grounded. In relation to Higher-Order Thinking Skills (HOTS), CL's ability to select efficient strategies, justify her steps, and connect multiple procedures (elimination and substitution) reflects the integration of analysis, evaluation, and strategic thinking.

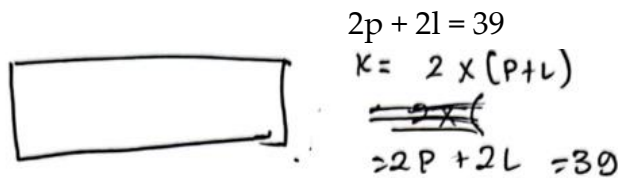
These findings indicate that HOTS develop through the coordination and restructuring of formal mathematical knowledge rather than through isolated procedural practice. Furthermore, CL's ability to restate and apply algorithmic patterns in a structured manner shows that her understanding had evolved into a coherent scheme, aligning with the theory of mathematical understanding growth, which emphasizes the formation of interconnected knowledge structures (Gulkilik et al., 2020). From an educational perspective, this suggests that mathematics instruction

should not only emphasize procedural mastery but also encourage students to reflect on their reasoning, compare alternative strategies, and justify their choices. Such practices can support students in developing a structured understanding and higher-order thinking simultaneously. Therefore, this finding signifies that the growth of mathematical understanding and higher-order thinking is a dynamic and integrated process, where the ability to coordinate, reflect, and generalize solution strategies becomes central to meaningful mathematical learning.

Structuring Level of CL

The following is the interview with CL.

CL : The perimeter, as I remember, is: $k = 2 \times (p+l)$; $k = 39$



The diagram shows a hand-drawn rectangle. To its right, the student has written the following work:

$$2p + 2l = 39$$

$$k = 2 \times (p+l)$$

The second equation is crossed out with a double line. Below it, the student has written:

$$= 2p + 2l = 39$$

P : How about the difference between length and width?

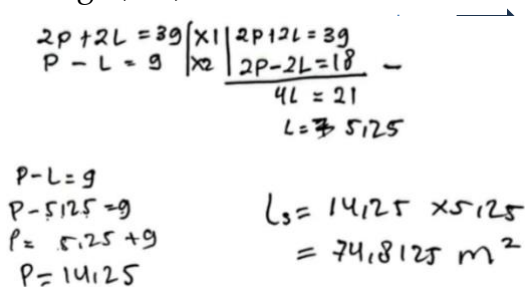
CL : Um... what is it, sir?

P : Suppose Budi has 2000 money and Amir has 1500 money. What is the difference between Budi's and Amir's money?

CL : 500 sir.

P : Please solve it.

CL : Alright, sir, like this:



The student's work shows the following steps:

$$\begin{array}{r|l} 2p + 2l = 39 & \times 1 \quad 2p + 2l = 39 \\ p - l = 9 & \times 2 \quad 2p - 2l = 18 \\ \hline & 4l = 21 \\ & l = 5,25 \end{array}$$

Below this, the student has written:

$$p - l = 9$$

$$p - 5,25 = 9$$

$$p = 5,25 + 9$$

$$p = 14,25$$

And for the area calculation:

$$L_s = 14,25 \times 5,25$$

$$= 74,8125 \text{ m}^2$$

P : Nice

Students' higher-order thinking skills in solving systems of linear equations in two variables (SPLDV) are strongly influenced by the pedagogical approaches employed during the learning process, as also reflected in this study on Exploring the Growth of a Junior High School Student's Mathematical Understanding and Higher-Order Thinking. The findings indicate that CL's ability to perform elimination and substitution procedures effectively is not solely a result of individual capability, but is closely linked to guided instructional support (scaffolding) that helped structure her reasoning when she got stuck. Previous studies have shown that approaches emphasizing conceptual understanding can significantly reduce students' errors in

applying algorithmic procedures (Diananda & Rahaju, 2023), while students' success in correctly eliminating variables and performing substitutions is associated with well-developed logical thinking (Alqhtani et al., 2023).

Furthermore, from a cognitive perspective, students equipped with effective problem-solving strategies are better able to manage the complexity of the variables involved in SPLDV tasks (Nari, 2023), which explains why CL's performance improved as her strategy became more organized and reflective. However, it is important to acknowledge that CL's progression was, to some extent, facilitated by the researcher's guiding questions, which directed her thinking processes, particularly through mechanisms such as folding back. This suggests that the observed growth of mathematical understanding and higher-order thinking may not be entirely autonomous, but rather co-constructed through interaction. From a theoretical standpoint, this supports the view that the growth of mathematical understanding and HOTS is socially mediated and develops through guided reflection and scaffolding, rather than emerging independently.

Consequently, the relationship between understanding growth and HOTS can be seen as dynamic and interactive, where structured support enables students to reorganize knowledge and engage in deeper reasoning. From an educational perspective, these findings imply that mathematics instruction should incorporate guided questioning, reflective activities, and opportunities for students to articulate and refine their thinking, while gradually reducing scaffolding to foster independent reasoning. Therefore, this study signifies that effective mathematics learning involves not only the development of procedural and conceptual knowledge, but also the careful design of instructional support that promotes the transition from guided to autonomous higher-order thinking.

CL's Folding Back in Mathematical Understanding Layers

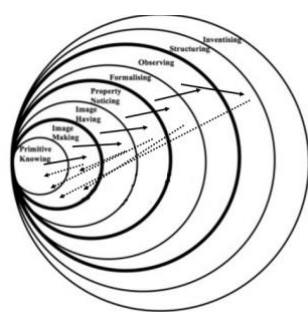


Figure 1. CL's Folding Back in Solving Mathematical Problems

Figure 1 illustrates CL's folding back activities at the level of mathematical understanding, indicating that CL tends to return to lower layers of understanding when faced with problems that cannot be solved directly. In this study, the folding back process is described through five phases: (1) in the first phase, CL was at the image making level and returned to primitive knowing to recall basic concepts; (2) in the second phase, CL at the image having level returned to image making; (3) in the

third phase, CL at the property noticing level returned again to primitive knowing; (4) in the fourth phase, CL at the formalising level returned to the image making layer; and (5) in the fifth phase, CL at the structuring level also returned to the image making layer.

This folding-back process enabled CL to rectify inadequate or incomplete understanding at certain levels by reorganizing previously held conceptual structures. Through this process, CL was able to reach higher levels of understanding by building or generating new images at previous levels when existing constructions were insufficient to solve the problem. Thus, the student demonstrated thinking abilities encompassing critical, logical, reflective, metacognitive, and creative thinking (King et al., 1998). The characteristics of student CL in the growth of mathematical understanding and stimulating higher-order thinking are presented in Table 1.

Table 1. CL's Characteristics in Mathematical Understanding Growth

Level/Layer of Mathematical Understanding	Description of Characteristics of Growth Mathematical Understanding and Higher-order Thinking
Primitive Knowing	Recognizes and understands the problem context, identifies given information, and forms initial connections, but reasoning remains concrete and tied to context. Serves as the foundation for higher-order thinking.
Image Making	Begins constructing mental representations by linking prior knowledge (PLDV) to new situations (SPLDV), though understanding is still exploratory and supported by scaffolding.
Image Having	Develops a stable understanding and applies concepts consistently (e.g., forming equivalent equations), showing emerging flexibility and early analytical thinking.
Property Noticing	Recognizes relationships and structures between concepts (e.g., elimination and substitution), indicating deeper conceptual understanding and analytical HOTS.
Formalising	Generalizes procedures into structured and consistent methods (algorithms), reflecting abstraction, logical reasoning, and justification of strategies.
Observing	Reflects on and coordinates formal knowledge, evaluates and refines solution strategies, demonstrating critical and evaluative thinking.
Structuring	Integrates and connects mathematical concepts across topics into a coherent system and applies reasoning logically to solve complex problems, representing advanced HOTS.

Table 1 shows the level of growth in mathematical understanding and higher-order thinking of student CL, which unfolds dynamically across the initial stages of learning (Pirie & Martin, 2000). Based on the Pirie–Kieren Theory framework, this growth begins with Primitive Knowing and progresses to Structuring. The Pirie–Kieren Theory in this study was used to reveal student activities and higher-order thinking,

in line with the demands of HOTS tasks that require students to critically evaluate information, draw conclusions, and make generalizations (Yee et al., 2016).

CONCLUSION

This study reveals that the growth of junior high school students' mathematical understanding of Systems of Linear Equations in Two Variables (SPLDV) occurs. This study reveals that the growth of mathematical understanding is a dynamic, recursive, and interconnected process that significantly contributes to the emergence of higher-order thinking skills (HOTS). The findings demonstrate that CL's progression from primitive knowing to structuring was not linear but involved iterative processes, such as folding back, in which prior knowledge was revisited and reorganized to achieve deeper conceptual understanding.

This indicates that higher-order thinking does not arise independently, but is gradually constructed through the refinement, integration, and coordination of conceptual knowledge across levels. A key contribution of this study lies in its synthesis of the Pirie–Kieren framework with HOTS, showing that each layer of understanding plays a distinct role in fostering analytical, evaluative, and flexible thinking. Furthermore, the findings highlight that CL's growth was partly influenced by guided instructional support, such as scaffolding, suggesting that the observed understanding was co-constructed rather than entirely autonomous. This insight emphasizes the importance of instructional design in facilitating students' cognitive growth, particularly through guided questioning, reflective activities, and opportunities to explore multiple solution strategies.

However, this study is limited by its focus on a single subject, which restricts the generalizability of the findings, as well as the potential influence of researcher intervention on the subject's thinking processes. Therefore, future research is recommended to involve a larger, more diverse sample and to examine the balance between scaffolding and independent learning in fostering higher-order thinking. Additionally, further studies could explore the application of the Pirie–Kieren framework in different mathematical topics and learning environments. From a broader perspective, this study implies that effective mathematics education should move beyond procedural instruction toward approaches that support conceptual understanding, reflective thinking, and the gradual development of independent reasoning.

Thus, the growth of mathematical understanding and higher-order thinking should be viewed as an integrated process, in which well-designed instructional support plays a crucial role in enabling students to achieve meaningful and sustainable learning, dynamically and non-linearly, according to the Pirie–Kieren theoretical framework, from the primitive knowing level to structuring. This process is characterized by students' ability to construct mathematical models, manipulate concepts, and coordinate formal procedures systematically, which does not always

progress forward but involves folding back activities to strengthen understanding when encountering difficulties. This layered growth of understanding directly contributes to the emergence of higher-order thinking skills (HOTS), such as the ability to analyze, evaluate, reflect on strategies, and generalize concepts to other contexts. Thus, mathematics learning is designed around layers of understanding, and facilitating the folding-back process is essential for optimally developing students' higher-order thinking skills.

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